DESCRIPTION OF ALL COMPLEX GEODESICS IN THE SYMMETRIZED BIDISC

Peter Pflug & Włodzimierz Zwonek

Abstract. In the paper we find effective formulas for the complex geodesics in the symmetrized bidisc.

1. Introduction. In the paper we deal with the problems arising while the study of the so-called symmetrized bidisc. The study of this domain was initiated by J. Agler, F. B. Yeh and N. J. Young (see e.g. [Agl-You 1], [Agl-Yeh-You]). Their interest came from the spectral theory. However, they found out that some of the results obtained for the symmetrized bidisc are interesting from the viewpoint of the geometric function theory, especially from the theory of holomorphically invariant functions. In particular, the symmetrized bidisc is an example of a bounded pseudoconvex domain, which is not biholomorphic to any convex domain and on which the Lempert function and the Carathéodory distance coincide (see [Agl-You 2], [Cos 2]). Therefore, it is the first example showing that the Lempert theorem (i.e. the coincidence of all holomorphically invariant functions) may extend non-trivially onto a larger class of domains than that of convex domains. Note that even more is known, namely, the symmetrized bidisc is not the exhaustion of domains biholomorphic to convex domains – see [Edi 2].

As a consequence of the above mentioned equality a natural notion of a complex geodesic arise (see e.g. [Jar-Pfl], the definition follows). The main aim of the paper is to give a description of all complex geodesics of the symmetrized bidisc (see Theorem 2). As a corollary we get an interesting phenomenon of the symmetrized bidisc; namely, the complex geodesics in the symmetrized bidisc extend holomorphically through the boundary and, unlike in all known cases, their boundaries lie in a very thin part of the boundary of the symmetrized bidisc (see Corollary 3).

Let us note that the description of complex geodesics together with the proof of their uniqueness was announced in [Agl-You 2].

2. Basic notions and definitions in the theory of holomorphically invariant functions. Let us start with basic definitions of the holomorphically invariant functions. For more references on this theory we refer the interested reader to consult [Jar-Pfl] and [Kob].

The research started while the second author was visiting Universität Oldenburg in the summer 2003 (supported by DAAD) and was finished while the stay in November 2003 (supported by DFG Project: PF 227/8-1). The second author was also supported by the KBN Grant No 5 P03A 033 21.

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$ and let p denote the Poincaré distance on $\mathbb{D}$. For a domain $D \subset \mathbb{C}^n$ and for points $w, z \in D$ we define

$$c_D(w,z) := \sup\{p(f(w), f(z)) : f \in \mathcal{O}(D, \mathbb{D})\};$$
$$\tilde{k}_D(w,z) := \inf\{p(\lambda_1, \lambda_2) : \text{ there is } f \in \mathcal{O}(\mathbb{D}, D) \text{ with } f(\lambda_1) = w,\ f(\lambda_2) = z\};$$
$$k_D := \text{the largest pseudodistance not larger than } \tilde{k}_D.$$

We call c_D (respectively, k_D) the *Carathéodory* (respectively, *Kobayashi*) *pseudodistance*. $\tilde{k}_D$ is called the *Lempert function*. The following simple inequalities are crucial:

$$c_D \le k_D \le \tilde{k}_D.$$

The important theorem of Lempert states that if D is biholomorphically equivalent to a convex domain then the inequalities above become equalities (see [Lem]), i.e.,

for any domain D biholomorphic to a convex domain we have $c_D = \tilde{k}_D$.

A mapping $\varphi \in \mathcal{O}(\mathbb{D}, D)$ is called a $\tilde{k}_D$*-geodesic* for (w, z), $w \ne z$, if $\varphi(\lambda_1) = w$, $\varphi(\lambda_2) = z$ and $p(\lambda_1, \lambda_2) = \tilde{k}_D(w, z)$ for some $\lambda_1, \lambda_2 \in \mathbb{D}$. In other words, $\tilde{k}_D$-geodesics are those mappings for which the infimum in the definition of $\tilde{k}_D$ is attained.

If D is a taut domain (the domain D is *taut* if for any sequence $(\varphi_\nu)_{\nu=1}^\infty \subset \mathcal{O}(\mathbb{D}, D)$ there is a subsequence converging locally uniformly on $\mathbb{D}$ to $\varphi \in \mathcal{O}(\mathbb{D}, D)$ or there is a subsequence $(\varphi_{\nu_k})_{k=1}^\infty$ such that for any compact $L \subset D$ and for any compact $K \subset \mathbb{D}$ there is a $k_0 \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $k \ge k_0$ $\varphi_{\nu_k}(K) \cap L = \emptyset$) then for any $w \ne z$, $w, z \in D$ there is a $\tilde{k}_D$-geodesic for (w, z). It is simple to conclude from the Schwarz-Pick lemma that if $\varphi \in \mathcal{O}(\mathbb{D}, D)$, $w, z \in D$, $w \ne z$, $\varphi(\lambda_1^0) = w$, $\varphi(\lambda_2^0) = z$ and $p(\lambda_1^0, \lambda_2^0) = c_D(w, z)$ for some $\lambda_1^0, \lambda_2^0 \in \mathbb{D}$, then $c_D(\varphi(\lambda_1), \varphi(\lambda_2)) = \tilde{k}_D(\varphi(\lambda_1), \varphi(\lambda_2)) = p(\lambda_1, \lambda_2)$ for any $\lambda_1, \lambda_2 \in \mathbb{D}$. This observation leads us to the following definition. A mapping $\varphi \in \mathcal{O}(\mathbb{D}, D)$ is called *a complex geodesic* (in D) if

$$c_D(\varphi(\lambda_1), \varphi(\lambda_2)) = p(\lambda_1, \lambda_2),$$

for any $\lambda_1, \lambda_2 \in \mathbb{D}$. Certainly, if φ is a complex geodesic then $c_D(\varphi(\lambda_1), \varphi(\lambda_2)) = \tilde{k}_D(\varphi(\lambda_1), \varphi(\lambda_2))$, $\lambda_1, \lambda_2 \in \mathbb{D}$.

The problem of finding explicit formulas for complex geodesics (or $\tilde{k}_D$-geodesics) is, in general, very difficult. One of very few non-trivial examples for which the formulas for complex geodesics are known completely are convex complex ellipsoids (see [Jar-Pfl-Zei]) or more generally convex generalized pseudoellipsoids (see [Zwo]). Another example of a convex domain, where the complex geodesics are known is the so-called minimal ball (see [Pfl-Yous]). Without the assumption of convexity only necessary forms of $\tilde{k}_D$-geodesics are known (see [Edi 1] also [Pfl-Zwo]).

3. Basic properties of the symmetrized bidisc and statement of results. Now let us start the study of the symmetrized bidisc. Define

$$\pi : \mathbb{D} \ni (\lambda_1, \lambda_2) \mapsto (\lambda_1 + \lambda_2, \lambda_1\lambda_2) \in \mathbb{C}^2.$$

Put $G_2 := \pi(\mathbb{D}^2)$. It is easy to see that G_2 is a domain. We call the domain G_2 the *symmetrized bidisc*. It is easy to calculate that

$$G_2 = \left\{ (s,p) \in \mathbb{C}^2 : \max\left\{ \left| \frac{s \pm \sqrt{s^2 - 4p}}{2} \right| \right\} < 1 \right\}. \tag{1}$$

It follows from this description that G_2 is a bounded hyperconvex domain in $\mathbb{C}^2$ (we recall that the bounded domain $D \subset \mathbb{C}^n$ is *hyperconvex* if there is a continuous plurisubharmonic function $u : D \mapsto (-\infty, 0)$ such that $\lim_{D \ni z \to \partial D} u(z) = 0$). In particular, G_2 is taut, so for any pair of points $w, z \in D$, $w \neq z$ there is a $\tilde{k}_{G_2}$-geodesic for (w,z). Let us recall that there is a number of other possible descriptions of the symmetrized bidisc (see e.g. [Agl-You 1], [Agl-You 2] and [Agl-Yeh-You]). Nevertheless, since we make use only of the above mentioned descriptions, therefore we do not recall them here.

Note that $\pi : \mathbb{D}^2 \mapsto G_2$ is a proper holomorphic mapping with multiplicity 2 and $\pi : \mathbb{D}^2 \setminus \triangle \mapsto G_2 \setminus S$ is a holomorphic covering, $\triangle := \{(\lambda, \lambda) : \lambda \in \mathbb{D}\}$, $S := \{(2\lambda, \lambda^2) : \lambda \in \mathbb{D}\}$.

Our first aim is a description of $k_{\tilde{G}_2}$-geodesics for the points $((0,0),(s_0,p_0))$ with $(s_0,p_0) \neq (0,0)$. The result below may be treated as the generalization of the 'Schwarz Lemma' for the symmetrized bidisc as formulated in [Agl-You 1] (we give an alternative description of geodesics and the uniqueness of the solution of the extremal problem – i.e. the uniqueness of geodesics passing through $(0,0)$). Moreover, the description of the $\tilde{k}_{G_2}$-geodesics seems to us to be simpler and more natural than that in [Agl-You 1].

Theorem 1. *Let $\varphi : \mathbb{D} \mapsto G_2$ be a holomorphic mapping such that $\varphi(0) = (0,0)$, $\varphi(\sigma^2) = (s_0, p_0) \neq (0,0)$ for some $\sigma \in \mathbb{D}$. Then the following are equivalent:*

$$\varphi \text{ is a } \tilde{k}_{G_2}\text{-geodesic for } ((0,0),(s_0,p_0)); \tag{2}$$

$$\text{there is a Blaschke product of degree less than or equal to two} \tag{3}$$
$$\text{with } B(0) = 0 \text{ and } \varphi(\lambda) = (B(\sqrt{\lambda}) + B(-\sqrt{\lambda}), B(\sqrt{\lambda})B(-\sqrt{\lambda})),\ \lambda \in \mathbb{D}.$$

Moreover, any mapping satisfying (2) *(or* (3)*) is a complex geodesic.*

Additionally, the $\tilde{k}_{G_2}$-geodesics for $((0,0),(s_0,p_0))$ are unique (up to automorphisms of $\mathbb{D}$).

Remark. It is easy to see that the function φ defined in (3) is a holomorphic function on $\mathbb{D}$, $\varphi(0) = (0,0)$ and its image lies in G_2. Moreover, we may write down explicitly all the possible forms of φ satisfying (3). Namely, if $B(\lambda) = \tau\lambda$, $\lambda \in \mathbb{D}$, $|\tau| = 1$ then

$$\varphi(\lambda) = (0, -\tau^2\lambda),\ \lambda \in \mathbb{D}. \tag{4}$$

If $B(\lambda) := \tau\lambda\frac{\lambda-\alpha}{1-\bar{\alpha}\lambda}$, $\lambda \in \mathbb{D}$, $\tau \in \partial\mathbb{D}$, $\alpha \in \mathbb{D}$ then

$$\varphi(\lambda) = \left(\frac{2\tau\lambda(1-|\alpha|^2)}{1-\bar{\alpha}^2\lambda}, \tau^2\lambda\frac{\lambda-\alpha^2}{1-\bar{\alpha}^2\lambda}\right),\ \lambda \in \mathbb{D}. \tag{5}$$

As already mentioned, the Lempert function and the Carathéodory distance of the symmetrized bidisc coincide (see [Agl-You 2], [Cos 2]), i.e.

$$c_{G_2} = \tilde{k}_{G_2}.$$

This, together with the fact that G_2 is not biholomorphic to any convex domain (see [Cos 1]), makes the domain interesting from the viewpoint of the theory of holomorphically invariant functions. In the theorem below we shall make use of the above mentioned equality to give a description of all $\tilde{k}_{G_2}$-geodesics (or equivalently, complex geodesics) in G_2.

Before we formulate the result let us recall how we can define automorphisms of G_2, which map $(0,0)$ onto an arbitrary point of S (see [Agl-You 1]). Namely for the fixed $\alpha \in \mathbb{D}$ we define the automorphism of G_2 (mapping $(0,0)$ onto $(2\alpha, \alpha^2)$ as follows. Let $b_\alpha(\lambda) := \frac{\alpha-\lambda}{1-\bar{\alpha}\lambda}$, $\lambda \in \mathbb{D}$. Assume that $(s,p) = \pi(\lambda_1,\lambda_2) = \pi(\lambda_2,\lambda_1)$, where $(\lambda_1,\lambda_2) \in \mathbb{D}^2$. Define the automorphism B_α of G_2 as follows:

$$B_\alpha(s,p) := \pi(b_\alpha(\lambda_1), b_\alpha(\lambda_2)).$$

Therefore, in view of Theorem 1, the whole problem with the description of complex geodesics reduces to the characterization of geodesics omitting the set S (i.e. such that $\varphi(\mathbb{D}) \cap S = \emptyset$).

The description of geodesics follows.

Theorem 2. *Let $\varphi : \mathbb{D} \mapsto G_2$. Then φ is a complex geodesic if and only if one of two possibilities holds*

(i) *if $\varphi(\mathbb{D}) \cap S \neq \emptyset$, then $\varphi = \tilde{\varphi} \circ b_\alpha$, where $\alpha \in \mathbb{D}$,*

$$\tilde{\varphi}(\lambda) = \pi(B(\sqrt{\lambda}), B(-\sqrt{\lambda})),\ \lambda \in \mathbb{D},$$

and B is a Blaschke product of degree one or two;

(ii) *if $\varphi(\mathbb{D}) \cap S = \emptyset$, then $\varphi = \pi \circ f$, where $f = (f_1, f_2)$, $f_1, f_2 \in \operatorname{Aut}\mathbb{D}$ and $f_1 - f_2$ has no root in $\mathbb{D}$.*

As an immediate corollary of Theorem 2 we get the following interesting property of complex geodesics of the symmetrized bidisc.

Corollary 3. *Let $\varphi : \mathbb{D} \mapsto G_2$ be a complex geodesic. Then φ extends holomorphically onto a neighborhood of $\bar{\mathbb{D}}$ and $\varphi(\partial\mathbb{D}) \subset \pi(\partial\mathbb{D} \times \partial\mathbb{D})$.*

As a corollary of Theorem 1 (uniqueness), Theorem 2 and the existence of automorphisms B_α we get the following.

Corollary 4. *Let $\varphi : \mathbb{D} \mapsto G_2$ be a complex geodesic. Then one of the three possibilities occurs:*

(i) *$\varphi(\mathbb{D}) \cap S = \emptyset$;*

(ii) *$\#(\varphi(\mathbb{D}) \cap S) = 1$;*

(iii) *$\varphi(\mathbb{D}) = S$.*

Moreover, each of the possibilities (i)–(iii) *is satisfied by some complex geodesic.*

In the proofs of Theorems 1 and 2 essential role will be played by special rational functions among which we shall choose candidates for the extremals in the definition of the Carathéodory distance. Namely, put

$$F_0(s,p)=p,\ (s,p)\in G_2, \tag{6}$$

$$F_\omega(s,p):=\frac{2p-\omega s}{2-\bar\omega s},\ (s,p)\in G_2,\ \text{where } \omega\in\partial\mathbb{D}. \tag{7}$$

One may verify that $F_\omega(G_2)\subset\mathbb{D}$, $\omega\in\partial\mathbb{D}\cup\{0\}$ (see [Agl You]).

Now we go on to proofs of Theorems 1 and 2.

4. Proof of Theorem 1.

Proof of Theorem 1. First we show that any function φ defined in (3) is a complex geodesic. In particular, it will show the implication ((3) $\Longrightarrow$ (2)).

Let us fix φ as in (3). It is sufficient to find

$$F\in\mathcal{O}(G_2,\mathbb{D}) \text{ such that } F\circ\varphi=r,$$

where r is some rotation.

In fact, we shall show that the function F may be chosen as the one as in (6) or (7).

Let us consider the case when $B(\lambda)=\tau\lambda$, $\lambda\in\mathbb{D}$, where $\tau\in\partial\mathbb{D}$. Then $\varphi(\lambda)=(0,-\tau^2\lambda)$, $\lambda\in\mathbb{D}$, and then $F(s,p):=p$, $(s,p)\in D$, satisfies the desired property.

Therefore, we are left with the case $B(\lambda):=\tau\lambda\frac{\lambda-\alpha}{1-\bar\alpha\lambda}$, $\lambda\in\mathbb{D}$, where $\tau\in\partial\mathbb{D}$ and $\alpha\in\mathbb{D}$. In this case the function F will be as in (7).

First we consider the case $\alpha=0$, i.e. $\varphi(\lambda)=(2\tau\lambda,\tau^2\lambda^2)$, $\lambda\in\mathbb{D}$. It is sufficient to take $\omega=1$. So assume that $\alpha\neq0$. Recall that then

$$\varphi(\lambda)=\left(\frac{2\tau\lambda(1-|\alpha|^2)}{1-\bar\alpha^2\lambda},\tau^2\lambda\frac{\lambda-\alpha^2}{1-\bar\alpha^2\lambda}\right),\ \lambda\in\mathbb{D}.$$

Note that in view of the Schwarz Lemma it is sufficient to choose ω so that $|F(\varphi(\alpha^2))|=|\alpha|^2$ (because $F(\varphi(0))=0$ and $\alpha^2\neq0$). Let us calculate (substitute in the last equality $\omega:=\frac{|\alpha|^2}{\bar\alpha^2}\tau$)

$$|F(\varphi(\alpha^2))|=|F(\frac{2\tau\alpha^2}{1+|\alpha|^2},0)|=\frac{2|\alpha|^2}{(1+|\alpha|^2)\left|2-\bar\omega\frac{2\tau\alpha^2}{1+|\alpha|^2}\right|}=|\alpha|^2.$$

This completes the proof of the fact that any function φ defined by (3) is a complex geodesic.

Now we show the following:

for any $(s_0,p_0)\in D$, $(s_0,p_0)\neq(0,0)$ there is a mapping φ as defined in (3) such that φ goes through (s_0,p_0).

Let $(s_0,p_0)=\pi(t_1,t_2)$. Obviously $(t_1,t_2)\neq(0,0)$. It is sufficient to show that there is a finite Blaschke product of degree one or two and $\sigma\in\mathbb{D}\setminus\{0\}$ such that $B(0)=0$, $B(\sigma)=t_1$, and $B(-\sigma)=t_2$.

In the case $t_2=-t_1$ it is sufficient to take $B=\mathrm{id}$, $\sigma=t_1$. So assume that $t_2\neq -t_1$. It is suffices to find $\tau\in\partial\mathbb{D}$, $\sigma\in(0,1)$, and $\alpha\in\mathbb{D}$ such that

$$\tau\frac{\sigma-\alpha}{1-\bar\alpha\sigma}=\frac{t_1}{\sigma},\ \tau\frac{-\sigma-\alpha}{1+\bar\alpha\sigma}=\frac{t_2}{-\sigma}.$$

Now comparing the values $p(\sigma,-\sigma)$ and $p(\frac{t_1}{\sigma},\frac{t_2}{-\sigma})$ as $\sigma\in(\max\{|t_1|,|t_2|\},1)$ and making use of the Darboux property we get the existence of a $\sigma\in(\max\{|t_1|,|t_2|\},1)$ such that $p(\sigma,-\sigma)=p(\frac{t_1}{\sigma},\frac{t_2}{-\sigma})$. This easily implies the existence of τ and α as desired.

Now we prove the implication ((2) $\implies$ (3)) under the additional assumption that

$$\psi^{-1}(0)=\{0\},\ \text{where } \psi(\lambda):=\varphi_1^2(\lambda)-4\varphi_2(\lambda),\ \lambda\in\mathbb{D}. \tag{8}$$

Note that ψ is a bounded holomorphic function. We may write $\psi(\lambda)=\lambda^k\tilde\psi(\lambda)$, $\lambda\in\mathbb{D}$, where $\tilde\psi:\mathbb{D}\mapsto\mathbb{C}$ is a bounded holomorphic function and $\tilde\psi(\lambda)\neq 0$, $\lambda\in\mathbb{D}$.

Then, in view of the description of G_2 as in (1).

$$\left|\varphi_1(\lambda^2)+\lambda^k\sqrt{\tilde\psi(\lambda^2)}\right|<2,\ \lambda\in\mathbb{D},$$

where the root taken above is chosen arbitrarily so that the function B defined

$$B(\lambda):=\frac{\varphi_1(\lambda^2)+\lambda^k\sqrt{\tilde\psi(\lambda^2)}}{2},\ \lambda\in\mathbb{D},$$

is holomorphic on $\mathbb{D}$. Note that $B(0)=0$ and $|B|<1$ on $\mathbb{D}$. Then we may write

$$\lambda^k\sqrt{\tilde\psi(\lambda^2)}=2B(\lambda)-\varphi_1(\lambda^2),\lambda\in\mathbb{D}.$$

Taking the square in the last equality we get

$$\psi(\lambda^2)=\lambda^{2k}\tilde\psi(\lambda^2)=4B^2(\lambda)-4B(\lambda)\varphi_1(\lambda^2)+\varphi_1^2(\lambda^2),\ \lambda\in\mathbb{D}.$$

Consequently,

$$B(\lambda)\varphi_1(\lambda^2)-B^2(\lambda)=\varphi_2(\lambda^2),\ \lambda\in\mathbb{D}. \tag{9}$$

It follows from (9) that for any $\lambda\in\mathbb{D}$

$$B(\lambda)\varphi_1(\lambda^2)-B^2(\lambda)=\varphi_2(\lambda^2)=B(-\lambda)\varphi_1(\lambda^2)-B^2(-\lambda),\ \lambda\in\mathbb{D},$$

or

$$(B(\lambda)-B(-\lambda))(\varphi_1(\lambda^2)-(B(\lambda)+B(-\lambda)))=0,\ \lambda\in\mathbb{D}.$$

In view of the identity principle for holomorphic functions two cases are possible:

Case (I). $B(\lambda)=B(-\lambda)$, $\lambda\in\mathbb{D}$.

Case (II). $\varphi_1(\lambda^2)=B(\lambda)+B(-\lambda)$, $\lambda\in\mathbb{D}$.

In Case (I) we easily get the existence of a function $B_1\in\mathcal{O}(\mathbb{D},\mathbb{D})$ such that $B_1(\lambda^2)=B(\lambda)=\frac{\varphi_1(\lambda^2)+\lambda^k\sqrt{\tilde\psi(\lambda^2)}}{2}$, $\lambda\in\mathbb{D}$. Taking the other holomorphic branch in the formula for B_1 we may define analogously the function $B_2\in(\mathbb{D},\mathbb{D})$ such that

$$\varphi=\pi\circ(B_1,B_2). \tag{10}$$

Then it follows from our earlier considerations that there is some $F\in\mathcal{O}(G_2,\mathbb{D})$ such that $F(0,0)=0$ and $|F(\varphi(\lambda))|=|\lambda|$, where F is as in (6) or (7). If F is as in (6) then, in view of the Schwarz Lemma,

$$|F(\pi(B_1(\lambda),B_2(\lambda)))|=|B_1(\lambda)||B_2(\lambda)|\leq|\lambda|^2,\ \lambda\in\mathbb{D}$$

– contradiction.

In the second case there is some $\tau\in\partial\mathbb{D}$ such that

$$\frac{2B_1(\lambda)B_2(\lambda)-\omega(B_1(\lambda)+B_2(\lambda))}{2-\bar\omega(B_1(\lambda)+B_2(\lambda))}=\tau\lambda,\ \lambda\in\mathbb{D};$$

so $2B_1(\lambda)B_2(\lambda)-\omega(B_1(\lambda)+B_2(\lambda))=2\tau\lambda-\tau\lambda\bar\omega(B_1(\lambda)+B_2(\lambda))$, $\lambda\in\mathbb{D}$. Differentiating at 0 we get (recall that $B_1(0)=B_2(0)=0$)

$$-\omega(B_1'(0)+B_2'(0))=2\tau.$$

The last equality together with the Schwarz inequality applied to B_1 and B_2 shows that $B_1(\lambda)=-\bar\omega\tau\lambda=B_2(\lambda)$, $\lambda\in\mathbb{D}$, so $\varphi(\lambda)=(-2\bar\omega\tau\lambda,\bar\omega^2\tau^2\lambda^2)$, $\lambda\in\mathbb{D}$, which shows that φ is as in (4) (and contradicts (8)), which finishes the proof in this case.

In Case (II) we easily get that

$$\varphi_1(\lambda)=B(\sqrt{\lambda})+B(-\sqrt{\lambda}),\ \lambda\in\mathbb{D}.$$

Then in view of (9) we get that

$$\varphi_2(\lambda)=B(\sqrt{\lambda})B(-\sqrt{\lambda}),\ \lambda\in\mathbb{D}.$$

Therefore, we are left with the proof of the fact that B is a Blaschke product of degree less than or equal to 2.

Let $(s_0,p_0)=\pi(t_1,t_2)=\pi(t_2,t_1)$, where $t_1,t_2\in\mathbb{D}$. Let $\varphi(\sigma^2)=(s_0,p_0)$. The assumption that φ is a $\tilde k_D$-geodesic for $((0,0),(s_0,p_0))$ leads us to the following observation: $B(\sigma)=t_1$, $B(-\sigma)=t_2$ (or vice versa, which, however, may be dealt analogously). Moreover, the function B is extremal in the following sense: $B\in\mathcal{O}(\mathbb{D},\mathbb{D})$, $B(0)=0$, $B(\sigma)=t_1$, $B(\sigma_2)=t_2$ and there is no holomorphic function $f:\mathbb{D}\mapsto\mathbb{C}$ such that $f(\mathbb{D})\subset\subset\mathbb{D}$ and $f(0)=0$, $f(\sigma)=t_1$, $f(-\sigma)=t_2$). Therefore, (see e.g. [Edi 1]) B must be a Blaschke product of degree smaller than or equal to two. So the implication ((2) $\implies$ (3)) (under the assumption (8)) is finished.

Now we show that the assumption that φ is a $\tilde k_{G_2}$-geodesic and the inequality $\psi^{-1}(0)\neq\{0\}$ implies that $\varphi(\lambda)=(2\tau\lambda,\tau^2\lambda^2)$, $\lambda\in\mathbb{D}$, for some $\tau\in\partial\mathbb{D}$. In fact,

assume that $\psi(\lambda_0)=0$ for some $\lambda_0\in\mathbb{D}\setminus\{0\}$. Consequently, $\varphi_1^2(\lambda_0)=4\varphi_2(\lambda_0)$. It follows from our considerations that φ must be a complex geodesic and that there is an $F\in\mathcal{O}(G_2,\mathbb{D})$ such that $F\circ\varphi$ is a rotation, where F is as in (6) or (7). In the first case $|\varphi_2(\lambda)|=|\lambda|$, $\lambda\in\mathbb{D}$. But because of the Schwarz Lemma we have $|\varphi_1^2(\lambda)|\le 4|\lambda|^2$, which together with the equality $|\varphi_1^2(\lambda_0)|=4|\varphi_2(\lambda_0)|=4|\lambda_0|$ gives the contradiction. So assume that there are $\omega,\tau\in\partial\mathbb{D}$ such that

$$\tau\lambda=\frac{2\varphi_2(\lambda)-\omega\varphi_1(\lambda)}{2-\bar{\omega}\varphi_1(\lambda)}. \tag{12}$$

Then we get the following equality (substitute $\lambda=\lambda_0$ in (12))

$$\varphi_1^2(\lambda_0)+2\varphi_1(\lambda_0)(\bar{\omega}\tau\lambda_0-\omega)-4\tau\lambda_0=0.$$

Solving the above equality with respect to $\varphi_1(\lambda_0)$ (note that $|\varphi_1(\lambda_0)|<2$) we get that $\varphi(\lambda_0)=-2\bar{\omega}\tau\lambda_0$. Applying the Schwarz Lemma to $\varphi_1:\mathbb{D}\mapsto 2\mathbb{D}$ we get that $\varphi_1(\lambda)=-2\bar{\omega}\tau\lambda$, $\lambda\in\mathbb{D}$. Substituting this formula in (12) it follows that $\varphi_2(\lambda)=\bar{\omega}^2\tau^2\lambda^2$, $\lambda\in\mathbb{D}$, which shows that φ is as in (4).

To finish the proof of the theorem it is sufficient to show the uniqueness. Let φ be a $\tilde{k}_{G_2}$-geodesic for $((0,0),(s_0,p_0))$ and $\varphi(\sigma^2)=(s,p)=\pi(t_1,t_2)$. Then φ is given by some Blaschke product of degree less than or equal to two. Moreover, the Blaschke product is such that $B(0)=0$ and, without loss of generality, $B(\sigma)=t_1$, $B(-\sigma)=t_2$. But the values at three points (0, σ and $-\sigma$) determine uniquely such Blaschke products. This completes the proof of the uniqueness and the proof of the whole theorem. □

Remark. Note that in the proof of Theorem 1 we did not make use of the equality $\tilde{k}_{G_2}=c_{G_2}$. In fact, we even showed the equality

$$\tilde{k}_{G_2}((0,0),(s_0,p_0))=c_{G_2}((0,0),(s_0,p_0)) \text{ for any } (s_0,p_0)\in G_2.$$

Assuming the equality $\tilde{k}_{G_2}=c_{G_2}$ from the very beginning we may simplify the proof a little. However, without this result our proof is not much longer but much more self-contained.

5. Proof of Theorem 2. In this part of the paper we provide the proof of Theorem 2. In contrast to the proof of Theorem 1 we make use of the equality $\tilde{k}_{G_2}=c_{G_2}$. In particular, we know that the notions of $\tilde{k}_{G_2}$-geodesics and complex geodesics coincide in the symmetrized bidisc.

Proof of Theorem 2. Note that the composition $b_\beta\circ B$, where $\beta\in\mathbb{D}$ and B is a Blaschke product of degree one or two, is a Blaschke product of degree one or two. Therefore, in view of Theorem 1 and because of the existence of automorphisms B_α mapping $(0,0)$ to an arbitrarily chosen point from S, to prove Theorem 2 it is sufficient to study the mappings φ omitting S.

Since we need this fact, note that we already know that if φ is a complex geodesic in G_2 such that $\varphi(\mathbb{D})\cap S\neq\emptyset$ then $\varphi(\partial\mathbb{D})\subset\pi(\partial\mathbb{D}\times\partial\mathbb{D})$.

To finish the proof of Theorem 2 it is sufficient to show the following equivalence.

Let $\varphi:\mathbb{D}\mapsto G_2$, $\varphi(\mathbb{D})\cap S=\emptyset$. Then φ is a complex geodesic if and only if $\varphi=\pi\circ f$, where $f=(f_1,f_2)$, $f_1,f_2\in\operatorname{Aut}\mathbb{D}$ and f_1-f_2 has no root in $\mathbb{D}$.

($\Rightarrow$): Let φ be a complex geodesic such that $\varphi(\mathbb{D}) \cap S = \emptyset$. Then there is a holomorphic function $f : \mathbb{D} \mapsto \mathbb{D}^2$ such that $f(\mathbb{D}) \cap \triangle = \emptyset$ and $\varphi = \pi \circ f$. One may easily verify that f is $\tilde{k}_{\mathbb{D}^2}$-geodesic, so it is a complex geodesic in $\mathbb{D}^2$. It is sufficient to show that $f = (f_1, f_2)$, where $f_1, f_2 \in \operatorname{Aut}\mathbb{D}$. Since f is a complex geodesic we easily obtain that at least one of its components is from $\operatorname{Aut}\mathbb{D}$. Suppose that the other component is not from $\operatorname{Aut}\mathbb{D}$. Without loss of generality let $f_1 \in \operatorname{Aut}\mathbb{D}$ and $f_2 \notin \operatorname{Aut}\mathbb{D}$. Fix $\sigma \in (0,1)$. Then there is a function $\tilde{f}_2 \in \mathcal{O}(\bar{\mathbb{D}}, \mathbb{D})$ such that $\tilde{f}_2(0) = f_2(0)$, $\tilde{f}_2(\sigma) = f_2(\sigma)$ and $\tilde{f}_2(\mathbb{D}) \subset\subset \mathbb{D}$. Then $(f_1, \tilde{f}_2)$ is a $\tilde{k}_{\mathbb{D}^2}$-geodesic for $(f(0), f(\sigma))$ (and consequently, a complex geodesic) and $\pi \circ (f_1, \tilde{f}_2)$ is a complex geodesic in G_2. But $f_1 - \tilde{f}_2$ has one root in $\mathbb{D}$ (use Hurwitz theorem), so $\tilde{\varphi} := \pi \circ (f_1, \tilde{f}_2)$ intersects S, which however contradicts the description of complex geodesics passing through S because $\tilde{\varphi}(\partial\mathbb{D}) \not\subset \pi(\partial\mathbb{D} \times \partial\mathbb{D})$.

($\Leftarrow$): Let $\varphi = \pi \circ (f_1, f_2)$ be such that $f_1, f_2 \in \operatorname{Aut}\mathbb{D}$ and the equation

$$f_1(\lambda) = f_2(\lambda) \tag{13}$$

has no root for $\lambda \in \mathbb{D}$. Without loss of generality we may assume that $f_1(\lambda) = \lambda$, $\lambda \in \mathbb{D}$ and $f_2(\lambda) = \tau \frac{\lambda - \alpha}{1 - \bar{\alpha}\lambda}$, $\lambda \in \mathbb{D}$, where $\tau \in \partial\mathbb{D}$ and $\alpha \in \mathbb{D} \setminus \{0\}$. One may easily verify that the nonexistence of roots of the equation (13) is equivalent to the inequality

$$|1 - \tau| \leq 2|\alpha| \tag{14}$$

(see e.g. Lemma 4.2 in [Agl-McCar])).

We shall prove the existence of a mapping $F = F_\omega \in \mathcal{O}(G_2, \mathbb{D})$, where $\omega \in \partial\mathbb{D}$, such that

$$\lim_{\lambda \to 0} \frac{1}{|\lambda|} m(F \circ \varphi(0), F \circ \varphi(\lambda)) = 1, \tag{15}$$

where $m(\lambda_1, \lambda_2) := \left|\frac{\lambda_1 - \lambda_2}{1 - \bar{\lambda}_1 \lambda_2}\right|$, $\lambda_1, \lambda_2 \in \mathbb{D}$. The above equality implies that $F \circ \varphi$ is an automorphism of the unit disk and, consequently, it shows that φ is a complex geodesic.

Elementary calculation show that the limit in (15) equals

$$\frac{|(1 + \tau\alpha\bar{\omega})^2 + \tau(1 - |\alpha|^2)|}{2 + 2\operatorname{Re}(\bar{\omega}\tau\alpha)}.$$

The last expression is smaller than or equal to

$$\frac{|1 + \tau\alpha\bar{\omega}|^2 + 1 - |\alpha|^2}{2 + 2\operatorname{Re}(\bar{\omega}\tau\alpha)}$$

with the equality holding iff $\frac{(1+\tau\alpha\bar{\omega})^2}{\tau} > 0$. But the last expression equals 1. So we shall finish the proof if we show for any $\tau \in \partial\mathbb{D}$ satisfying (14) the existence of $\tilde{\omega} \in \partial\mathbb{D}$ such that $\frac{(1+\tilde{\omega}|\alpha|)^2}{\tau} > 0$. It is easy to verify that it is sufficient to show the existence of such an $\tilde{\omega}$ for $\tau \in \partial\mathbb{D}$ satisfying the equality $|1 - \tau| = 2|\alpha|$ or $1 - \cos\tau = 2|\alpha|^2$ (and then without loss of generality we may assume that $\sin\tau > 0$). So let $\tau = 1 - 2|\alpha|^2 + i\sqrt{1 - |\alpha|^2}2|\alpha|$. Define $\tilde{\omega} := -|\alpha| + i\sqrt{1 - |\alpha^2}$. Then we may easily verify that $\frac{(1+\tilde{\omega}|\alpha|)^2}{\tau} > 0$, which finishes the proof. $\square$

Remark. Note that a more effective description of automorphisms f_1 and f_2 coming out in the case (ii) is given with he help of the inequality (14). Although the condition (14) applies to the situation when $f_1 = \mathrm{id}$, it gives no restriction on generality of the characterization.

The proof of the equality $c_{G_2} = \tilde{k}_{G_2}$ given by Costara in [Cos 2] (unlike the one of Agler and Young in [Agl-You 2]) is purely function-theoretic. So all the results obtained in the paper may be proven by the function theoretic means.

Our method led us to an alternate (but certainly equivalent) description of complex geodesics in the symmetrized bidisc to the one of Agler, Costara, Yeh and Young. In particular, our description suggests a possible form of $\tilde{k}_{G_n}$-geodesics (or probably complex geodesics) for G_n, $n \geq 3$, where G_n denotes the higher dimensional analogue of the symmetrized bidisc defined as follows. $G_n := \pi_n(\mathbb{D}^n)$, where

$$\pi_n(\lambda_1, \dots, \lambda_n) = (\lambda_1 + \dots + \lambda_n, \sum_{1 \leq j < k \leq n} \lambda_j \lambda_k, \dots, \lambda_1 \cdot \dots \cdot \lambda_n),$$

$\lambda_j \in \mathbb{C}$, $j = 1, \dots, n$. Namely, the suggested form of the $\tilde{k}_{G_n}$-geodesics passing through $(0, \dots, 0)$ (and for our convenience mapping 0 into $(0, \dots, 0)$) is of the following form

$$\varphi(\lambda) = \pi_n \circ (B(\varepsilon_0 \sqrt[n]{\lambda}), \dots, B(\varepsilon_{n-1} \sqrt[n]{\lambda})),\ \lambda \in \mathbb{D},$$

where in the definition above B is any Blaschke product of degree less than or equal to n, $B(0) = 0$, $\sqrt[n]{1} = \{\varepsilon_0, \dots, \varepsilon_{n-1}\}$ and the root $\sqrt[n]{\lambda}$ is chosen arbitrarily. The main obstacle in the possible proof of the fact that these functions are actually $\tilde{k}_{G_n}$-geodesics (or even complex geodesics) is the lack of good candidates of functions maximalizing the expression in the definition of the Carathéodory distance, which would play the role of the functions F_ω from the two dimensional case.

References

[Agl-McCar] J. Agler, J. E. McCarthy, *Norm preserving extensions of holomorphic functions from subvarieties of the bidisk*, Annals of Mathematics **157** (2003), 289–312.

[Agl-Yeh-You] J. Agler, F. B. Yeh, *Realization of functions into the symmetrised bidisc*, preprint.

[Agl-You 1] J. Agler, N. J. Young, *A Schwarz Lemma for the symmetrized bidisc*, Bull. London Math. Soc. **33** (2001), 195–210.

[Agl-You 2] J. Agler, N. J. Young, *The hyperbolic geometry of the symmetrized bidisc*, preprint.

[Cos 1] C. Costara, *The symmetrized bidisc as a counterexample to the converse of Lempert's theorem*, Bull. London Math. Soc. (to appear).

[Cos 2] C. Costara, *Dissertation*, Université Laval.

[Edi 1] A. Edigarian, *On extremal mappings in complex ellipsoids*, Ann. Pol. Math. **62** (1995), 83–96.

[Edi 2] A. Edigarian, *private communication (October 2003).*

[Jar-Pfl] M.Jarnicki & P.Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, 1993.

[Jar-Pfl-Zei] M. Jarnicki, P. Pflug & R. Zeinstra, *Geodesics for convex complex ellipsoids*, Ann. d. Sc. Norm. Sup. di Pisa **XX Fasc 4** (1993), 535–543.

[Kob] S. Kobayashi, *Hyperbolic Complex Spaces*, Grundlehren der mathematischen Wissenschaften, Springer, vol 318, 1998.

[Lem] L.Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), 427-479.

[Pfl-Yous] P. Pflug, E. H. Youssfi, *Complex geodesics of the minimal ball in* $\mathbb{C}^n$, preprint.

[Pfl-Zwo] P. Pflug & W. Zwonek, *The Kobayashi metric for non-convex complex ellipsoids*, Complex Variables **29** (1996), 59-71.

[Zwo] W. Zwonek, *Effective formulas for complex geodesics in generalized pseudoellipsoids with applications*, Ann. Pol. Math. **61** (1995), 261–294.

Carl von Ossietzky Universität Oldenburg, Institut für Mathematik, Postfach 2503, D-26111 Oldenburg, Germany

E-mail address: pflug@mathematik.uni-oldenburg.de

Uniwersytet Jagielloński, Instytut Matematyki, Reymonta 4, 30-059 Kraków, Poland

E-mail address: zwonek@im.uj.ed.pl